Counting Baxter Matrices

George Spahn†

†*Department of Mathematics, Rutgers University (New Brunswick)*
Email: gs828 at math dot rutgers dot edu

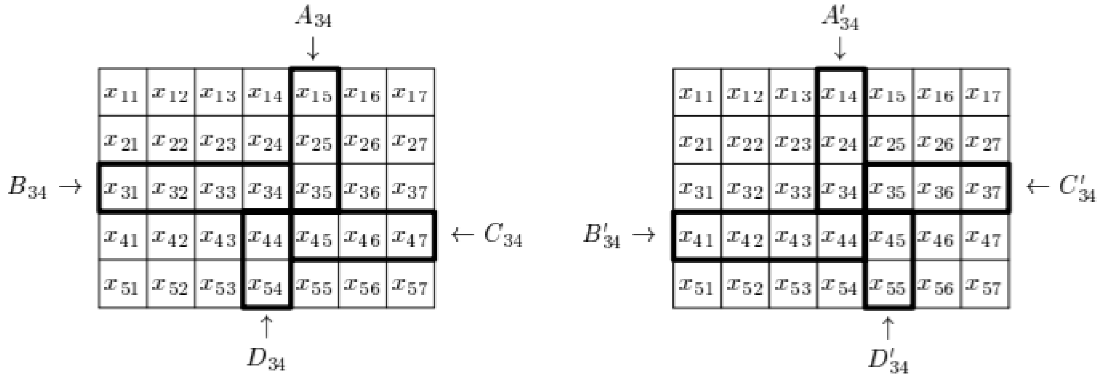
ABSTRACT: Donald Knuth recently introduced the notion of a Baxter matrix, generalizing Baxter permutations. We show that for fixed number of rows, r, the number of Baxter matrices with r rows and k columns eventually satisfies a polynomial in k of degree $2r-2$. We also give a proof of Knuth's conjecture that the number of 1's in a $r \times k$ Baxter matrix is less than $r+k$.

1. Introduction

Donald Knuth defines a Baxter matrix as an $m \times n$ matrix of 0's and 1's that satisfy 4 conditions [Kn21].

1. Each row contains a 1.
2. Each column contains a 1.
3. Each clockwise pinwheel contains a segment of all 0's.
4. Each counter-clockwise pinwheel contains a segment of all 0's.

Below is a diagram of the four segments of a clockwise pinwheel (left) and counter-clockwise pinwheel (right). The center of a pinwheel is allowed to be anywhere inside the matrix. Thus there are $(m-1)(n-1)$ pinwheels of each direction that must be satisfied. For the pinwheel on the left to be satisfied, at least one out of A_{34}, B_{34}, C_{34}, or D_{34} must contain only 0's. We refer the reader to Knuth's paper for the formal definition (the diagram has also been copied from there). `https://cs.stanford.edu/~knuth/papers/baxter-matrices.pdf`



2. A Finite State Automaton for Baxter Matrices with r rows

Let's construct a finite state machine for determining whether a 0-1 matrix with 2 rows is a Baxter Matrix. The symbols that our machine should recognize as input should be the possible columns in the matrix. There are 4 possible columns in a 0-1 matrix with 2 rows, $[0,0]^T$, $[0,1]^T$, $[1,0]^T$, and $[1,1]^T$. We can ignore the column $[0,0]^T$ because any column in a baxter matrix must not be all zeros.

As we move through the columns of our input matrix, our machine will keep track one of the following information for each row: whether the row is all 0's up to this point (so that it can be used to satisfy future pinwheels) and whether the row must be all 0's in the future (because a pinwheel from earlier is depending on it). Thus each row can be in one of 4 possible states; we will refer to them as the 4 rowstates:

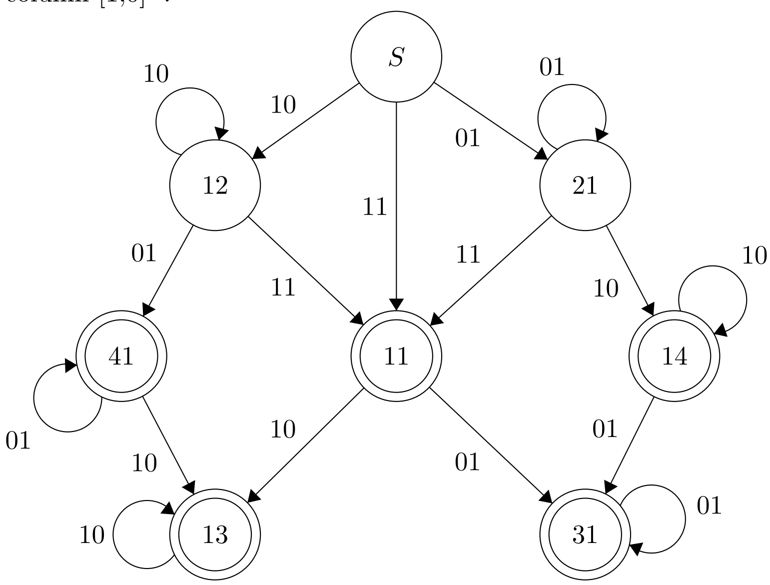
1. This row has a 1 in the most recent column.
2. This row only contains 0's.
3. This row must only contain 0's for the rest of the columns.
4. This row had a 0 in the most recent column but does not fit 2. or 3.

Our machine will have 16 states, one for each ordered pair of rowstates. We next remove the states which have all rows in rowstates 2, 3, and 4. This is because if all rows had a 0 most recently, then the corresponding column contains only 0's, and the matrix cannot be Baxter. This leaves us with $16 - 9 = 7$ states. I claim that the information contained in such a state is enough to determine which columns can come next. For each possible pinwheel with center between our most recent column and the proposed next column, we can mostly check whether it is satisfied with the information stored. The information about whether the vertical strips are all 0's is known because we store the exact contents of the most recent column in the state. The horizontal strip going left is available if and only if the corresponding row is in rowstate 2. The only thing we don't know yet is whether a horizontal strip going off to the right is all 0's, but if a pinwheel requires it to be so, we can set the rowstate of the corresponding row to 3, and keep track of it for later. We note some other basic constraints:

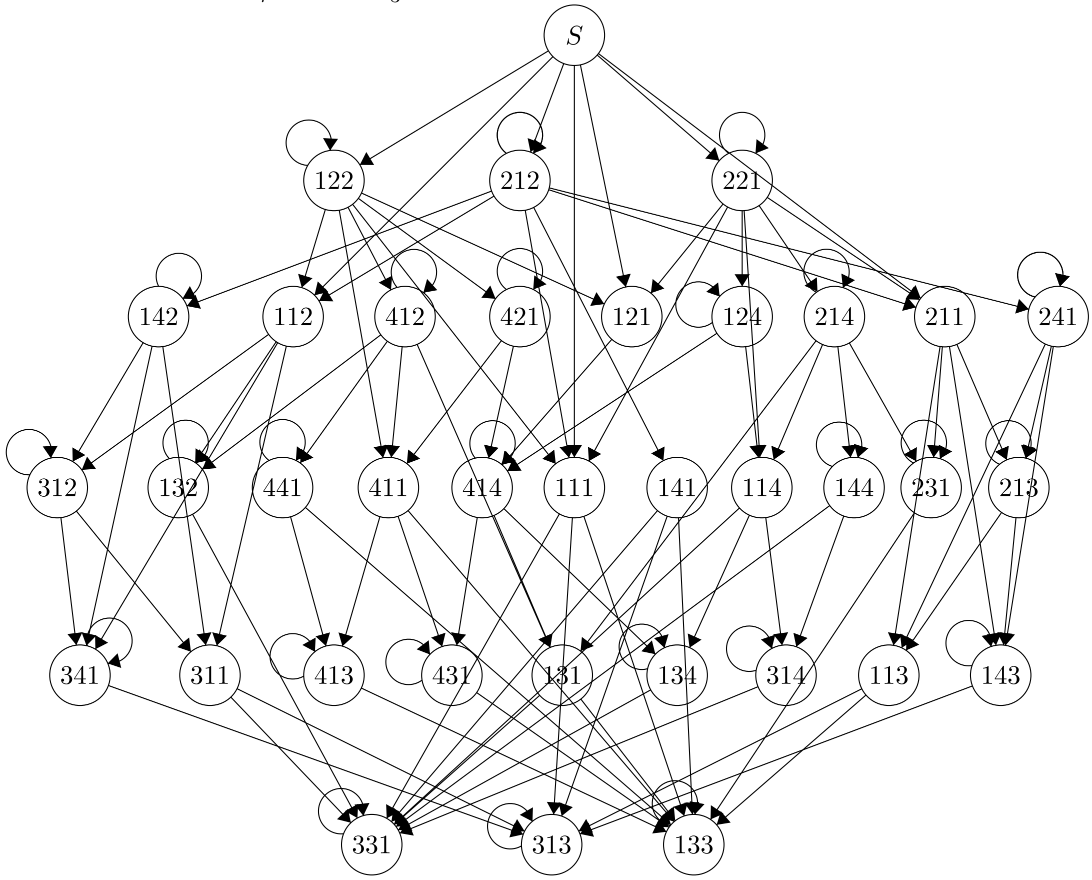
- Once a row is in rowstate 3 it can never leave rowstate 3.
- Once a row leaves rowstate 2 it can never come back to rowstate 2.
- A row cannot transition from 2 to 3 directly, or else it will be all zeros.

As the automaton proceeds reading columns, it checks whether each new pinwheel can be satisfied. The center of the new pinwheels is between the most recent column and proposed next column. When it encounters such a pinwheel that cannot be satisfied, it can immediately reject the sequence of columns. If a pinwheel cannot be satisfied given the first j columns, there is no Baxter matrix that begins with those first j columns. Similarly, if a pinwheel is satisfied given the first j columns, we do not need to keep track of that pinwheel any longer. It will still be satisfied in any Baxter matrix with those first j columns provided we keep track of which rows must be all 0's in the future.

We additionally add a start state that transitions to all states that have each row in either rowstate 1 or rowstate 2 (rowstates 3 and 4 cannot be reached using only a single column). We designate all states that have no rows in rowstate 2 as accept states (2 must be excluded so that no row of the final matrix is all 0's). Enforcing all of the rules we have described so far yields the following automaton. The state label 12 indicates that the first row is in rowstate 1 and the second row is in rowstate 2. The transition label 10 indicates the column $[1,0]^T$.



We can do this same process for any fixed number of rows. There will be 2^r symbols, and $4^r - 3^r$ states. Let's call this automaton A_r Here is A_3:



I have drawn the automata like this to motivate the following definition.

Definition 2.1. *The **depth** of a state is number of 1's plus the number of 4's plus twice the number of 3's that can be found in the rowstates of the rows. Let $d(s)$ denote the depth.*

Lemma 2.1. *For a fixed number of rows, r, any transition in A_r must either be a self arrow or increase depth.*

Proof. Consider an arbitrary transition. The two states in the transition, let's call them state 1 and state 2, fully specify the contents of two consecutive columns of the matrix, let's call them columns 1 and 2 respectively. If a row in the first state is in rowstate 3, it must be in rowstate 3 in state 2 as well. Similarly if a row in the second state is in rowstate 2, it must be in rowstate 2 in the first state as well. Therefore the contribution to depth from a single row cannot decrease. The total depth is a computed as a sum of contributions from each row, so it follows that the total depth cannot decrease. To show that if the depth remains the same, we must have a self transition, we first note that each individual row's contribution in a depth preserving transition must not increase depth. Otherwise, if such a row did increase its contribution, then we can apply the above reasoning to the other rows, and conclude that the total depth must increase. Therefore all 2's in the first state must remain 2's, and all 3's in the second state must have come from 3's in the first state.

We know that both columns must contain a 1 somewhere. Say the 1 in the first column is in row i. It cannot be the case that the only 1 in the second column is in row i, otherwise we would have a self arrow. Therefore there exists a column $j \neq i$ so that the second column contains a 1 in row j. Let's assume $j < i$, and note that the other case will be covered by symmetry. The clockwise pinwheel with center between the columns and directly below row j indicates that row $j + 1$ must end up in rowstate 3. Since the depth is preserved it must have started in rowstate 3 as well, which means the next clockwise pinwheel below that will require row $j + 2$ to end up in rowstate 3 as well. This pattern continues all the way down to row i where we start in rowstate 1 and end in rowstate 3, which necessarily increases the depth.

In fact we have shown something stronger: whenever there exists 1's in consecutive columns that are not in the same row, the corresponding transition must increase depth. Thus a self arrow can only emerge from states that contain a single 1.

□

Suppose we fix the number of rows, r, and want to count the number of Baxter matrices with k columns. They are in bijection with paths of length k on our state transition graph. If we ignore the self arrows, the lemma shows that there are only finitely many possibilities. We can classify all matrices into finitely many classes according to the path it takes with all the self arrows removed.

How many k column matrices are there of a particular class? Suppose the path corresponding to our class has length l and has m nodes with self arrows on it. Notice that the minimum possible depth of a state is 0 at the start state, and the maximum possible depth is $2r-1$. Thus $l \leq 2r-1$, and therefore $m \leq 2r-1$ as well. How many k column matrices are there of this class? We just have to choose how many self arrows to put at each node, so that the total path length is k. This is equivalent to choosing some natural numbers that sum to k - (the length of the path), one for each node with a self arrow on the given path. The number of ways to do this is a polynomial in k of degree $m-1$.

Now for each possible path without self arrows we will get a polynomial in k of degree at most $2r-2$, that counts the number of matrices with k columns for that path. The total number of matrices with r rows and k columns will be the sum over all these polynomials, which will again be a polynomial of degree at most $2r-2$. The only small caveat is that for small values of k we must take the maximum of each polynomial and 0, so some paths cannot contribute negatively. Summarizing:

Theorem 2.1. *For a fixed number of rows, r, the number of Baxter matrices with r rows and k columns eventually satisfies a polynomial in k of degree $2r-2$.*

Computing the polynomial for a fixed r is straightforward once the transition graph has been constructed. I have code that constructs the graph and computes the corresponding polynomial using linear algebra. The code completes instantly for $r \leq 5$ and within a couple minutes for $r = 6$.

rows	formula	works for
2	$k^2 + 3k - 4$	k ≥ 2
3	$(1/3)k^4 + 3k^3 - (16/3)k^2 + 2k + 3$	k ≥ 3
4	$(1/18)k^6 + (21/20)k^5 - (5/18)k^4 - (151/12)k^3 + (443/9)k^2 - (1012/15)k + 28$	k ≥ 4
5	$(23/4032)k^8 + (937/5040)k^7 + (853/1440)k^6 + \dots$	k ≥ 5
6	$(361/907200)k^{10} + (403/20160)k^9 + (5177/30240)k^8 + \dots$	k ≥ 6

3. Resolving one of Knuth's conjectures

Donald Knuth conjectured that any $m \times n$ Baxter matrix has fewer than $m+n$ 1's. We know from the definition of Baxter matrices that each column must contain at least one 1. We say that any column containing two or more 1's contains extra 1's.

Theorem 3.1. *The number of extra 1's in a Baxter matrix with r rows is less than r.*

To prove this, we will use the following lemma.

Lemma 3.1. *The total number of extra 1's that appear in two consecutive columns is at most the change in depth of the corresponding state transition in A_r.*

For now let's assume the lemma is true. Suppose M is some Baxter matrix with r rows and k columns. Let p be its corresponding path in A_r, and let T be the set of transitions in p. Let t^* be the final state in p. If we use the fact that the start state does not contain any extra 1's and assume that t^* does not contain any extra 1's, we get that

$$(\text{ of extra 1's in } M) = \frac{1}{2}\left(\sum_{\tau \in T} (\text{ of extra 1's in the columns associated with } \tau)\right) \tag{1}$$

If t^* does contain extra 1's, then we modify p to contain an extra transition from t^* to a state of maximal depth (regardless of whether such a transition exists in A_r). Note that any state of maximal depth must not contain any additional 1's. And additionally this transition satisfies Lemma 3 because for each extra 1 that appears in t^*, the maximum depth that t^* could be is reduced by 1. Now we apply the Lemma 3 to the (potentially modified) p.

$$\begin{aligned}
(\text{ of extra 1's in } M) &= \frac{1}{2}\left(\sum_{\tau \in T}(\text{ of extra 1's in the columns associated with } \tau)\right) && (2)\\
&\leq \frac{1}{2}\left(\sum_{\tau \in T}(\text{depth increase of } \tau)\right) && (3)\\
&\leq \frac{1}{2}(2r-1) && (4)\\
&< r && (5)
\end{aligned}$$

Now for a proof of Lemma 3:

Proof. Consider 2 consecutive columns. Let i be the minimum row which contains a 1 in the first column, j be the minimum row that contains a 1 in the second column, k be the maximum row which contains a 1 in the first column, and l be the maximum row that contains a 1 in the second column. We can assume $i \leq j$, for if not the proof is symmetric. We will consider 3 cases.

Case 1: $i \leq k \leq j \leq l$. We will show that each extra 1 in the first column forces the depth to increase by at least one. If $i = k$ then there are no extra 1's so there is nothing to show. So we have $i < k \leq l$. Let's look at the counterclockwise pinwheel above row k. The only way to satisfy it is if row $k-1$ ends up in rowstate 3. We can then look at the counterclockwise pinwheel directly above that one, and conclude that row $k-2$ must end up in rowstate 3 as well. We get that each row from $k-1$ up to i ends up in rowstate 3, so we have an increase in depth for each 1 in the first column from rows i to $k-1$, which is an increase in depth by 1 for each extra 1 in the first column. We also get that $j \geq k$. Now look at the counterclockwise pinwheel below row j. It forces row $j+1$ to start in rowstate 2. Applying the same process, all rows from row $j+1$ to l must start in rowstate 2. This gives an increase in depth for each 1 in the second column between row $j+1$ and l.

If we are not in case 1, then it must be the case that $k > j$.

Case 2: $i \leq j = l < k$. We will show that for each 1 that is not on row j, there must be an increase in depth. If there is a 1 above row j, then the counterclockwise pinwheel directly above row j can only be satisfied if row $j-1$ ends up in rowstate 3. Continuing to look at the counterclockwise pinwheels upward, all rows must end up in rowstate 3 from row $j-1$ up to row i. Thus all 1's in those rows cause there to be transitions from rowstate 1 into rowstate 3, causing an increase in depth. Similarly the clockwise pinwheel directly below row j requires row $j+1$ to end up in rowstate 3, and all rows below it down to row k as well. Thus each 1 below row j also causes an increase in depth.

Case 3: $i \leq j < l$ and $k > j$. It is not hard to show that in this case $i < j$ and there are no 1's in the first column on rows j through l. Let i' be the maximum row that contains a 1 that is less than j, and k' be the minimum row that contains a 1 that is greater than l. We will show that each 1 except for the 1's in rows i' and k' cause an increase in depth. By the same reasoning that we applied in case 1, there must be an increase in depth for each 1 above row i' and for each 1 below row k'. If there are at least two 1's in the second column, then each of those 1's must have transitioned from rowstate 2 in the first column, causing an increase in depth. If there is a single 1, then we can look at the clockwise pinwheel below it to conclude it either transitioned from rowstate 2 or the next row ends up in rowstate 3. In the first case we are done and in the second case we can progress the rowstate 3's down until we get that row k' must end up in rowstate 3.

□

Note that the number of extra 1's in a Baxter matrix is completely determined by its path in A_r. Also Lemma 3 shows that a self-arrow never can produce extra 1's. Therefore the self arrows in a path have no effect on the number of extra 1's. If we want to compute the number of $r \times k$ Baxter matrices with t 1's for some $k \leq t < k+r$, we can use the same method that we used to count them before, except only add the polynomials for paths without self arrows that add the appropriate amount of extra 1's. Here are some results:

$r = 3$, correct for $k \geq 3$

extra 1's	total weight	formula
0	k	$(1/3)k^4 - k^3 + (2/3)k^2$
1	k+1	$4k^3 - 12k^2 + 15k - 8$
2	k+2	$6k^2 - 13k + 11$

$r = 4$, correct for $k \geq 4$

extra 1's	total weight	formula
0	k	$(1/18)k^6 - (3/10)k^5 + (2/9)k^4 + (3/2)k^3 - (77/18)k^2 + (24/5)k - 2$
1	k+1	$(27/20)k^5 - (47/6)k^4 + (235/12)k^3 - (157/6)k^2 + (226/15)k$
2	k+2	$(22/3)k^4 - (121/3)k^3 + (335/3)k^2 - (500/3)k + 106$
3	k+3	$(20/3)k^3 - 32k^2 + (238/3)k - 76$

4. References

[Kn21] D.E.Knuth, Baxter matrices, http://cs.stanford.edu/ knuth/papers/baxter-matrices.pdf